Speed Round

LMT Fall 2024

December 14, 2024

1. **[6]** Find the value of
$$(2+0+2+4)+\left(2^0+2^4\right)+\left(2^{0^{2^4}}\right).$$

2. **[6]** The angles in triangle ABC are such that $\angle A$, $\angle B$, $\angle C$ form an arithmetic progression in that order. Find the measure of $\angle B$, in degrees.

3. **[6]** High schoolers chew a lot of gum. At the supermarket, 15 packs of 14 sticks of gum costs \$10. If 1400 high schoolers chew 3 sticks of gum per day, find the total number of dollars spent by these high schoolers on gum per week.

4. **[6]** Define $x \star y$ to be $xy \cdot \min(x, y)$ and $x \diamond y$ to be $xy \cdot \max(x, y)$. Suppose $ab = 4$. Find the value of
$$(a \star b) \cdot (a \diamond b).$$

5. **[6]** Find the area of the quadrilateral with vertices at $(0,0), (2,0), (20,24), (0,2)$ in that order.

6. **[6]** Danyang is doing math. He starts to draw an isosceles triangle, but only manages to draws an angle of 70° before he has to leave for recess. Find the sum of all possible values for the smallest angle in Danyang's triangle.

7. **[6]** Find the sum of the distinct prime factors of 512512.

8. **[6]** The LHS Math Team is doing Karaoke. William sings every song, David sings every other song, Peter sings every third song, and Muztaba sings every fourth song. If they sing 600 songs, find the average number of people singing each song.

9. **[6]** Find the median of the positive divisors of $6^4 - 1$.

10. **[6]** Today is 12/14/24, which is of the form $ab/ac/bc$ for not necessarily distinct digits a, b, and c. Find the number of other dates in the 21st century that can also be written in this form.

11. **[6]** Let x and y be real numbers such that
$$x + \frac{1}{y} = 20 \text{ and } y + \frac{1}{x} = 24.$$
Find $\frac{x}{y}$.

12. **[6]** Call a number *orz* if it is a positive integer less than 2024. Call a number *admitting* if it can be expressed as $a^2 - 1$ where a is a positive integer. Finally call a number *muztaba* if it has exactly 4 positive integer factors. Find the number of *muztaba admitting orz* numbers.

13. **[6]** Some math team members decide to study at Cary Library after school. They walk 6 blocks north, then 8 blocks west to get there. If they walk n blocks east from the library, they can buy boba from CoCo's. If CoCo's is the same distance from school as it is from the library, find n.

14. **[6]** Isabella assigns a distinct integer from 1 to 6 to each row and column of a 3×3 grid. In each entry, she writes either the sum or the product of the values assigned to the corresponding row and column. Find the maximum possible value of the sum of all entries in the grid.

15. **[6]** Find the value of $1 \cdot 2 \cdot 3 \cdot 4 + 2 \cdot 3 \cdot 4 \cdot 5 + \cdots + 6 \cdot 7 \cdot 8 \cdot 9$.

16. **[6]** Let $ZHAO$ be a square with area 2024. Let X be the center of this square and let C, D, E, K be the centroids of XZH, XHA, XAO, and XOZ, respectively. Find $[ZHAO]+[CZHAO]+[DZHAO]+[EZHAO]+[KZHAO]$.

    (Here $[\mathscr{P}]$ denotes the area of the polygon $\mathscr{P}$.)

17. **[6]** For positive integers x, let

$$f(x)=\begin{cases}\frac{f\left(\frac{x}{2}\right)}{2} & \text{if } x \text{ is even,}\\ 2^{-x} & \text{if } x \text{ is odd.}\end{cases}$$

    Find $f(1)+f(2)+f(3)+\dots$.

18. **[6]** Find the number of ways to split the numbers from 1 to 12 into 4 non-intersecting sets of size 3 such that each set has sum divisible by 3.

19. **[6]** Let $P(n)$ denote the product of digits of n. Find the number of positive integers $n \le 2024$ where $P(n)$ is divisible by n.

20. **[6]** Henry places some rooks and some kings in distinct cells of a 2×8 grid such that no two rooks attack each other and no two kings attack each other. Find the maximum possible number of pieces on the board.

    (Two rooks *attack* each other if they are in the same row or column and no pieces are between them. Two kings attack each other if their cells share a vertex.)

21. **[6]** Let ABC be a triangle with $\angle ABC = 90^\circ$. Let D and E be the feet from B and C to the median from A, respectively. Suppose $DE=4$ and $CD=5$. Find the area of ABC.

22. **[6]** Chris has a list of 5 distinct numbers and every minute he independently and uniformly at random swaps a pair of them. Find the probability that after 4 minutes the order of the list is the same as the original list.

23. **[6]** Circles ω_1 and ω_2 intersect at points X and Y. The common external tangent of the two circles closer to X intersects ω_1 and ω_2 at A and B, respectively. Given that $AB=6$, the radius of ω_1 is 3, and AY is tangent to ω_2, find XY^2.

24. **[6]** Find the number of positive integers x that satisfy

$$\left\lfloor \frac{2024}{\left\lfloor \frac{2024}{x} \right\rfloor} \right\rfloor = x.$$

25. **[6]** Let a_n be a sequence such that $a_1=1$, $a_2=1$, and $a_{n+2}=\frac{a_{n+1}a_n}{a_{n+1}+a_n}$. Find the value of

$$\sum_{n=1}^{\infty}\frac{1}{a_n 3^n}.$$